Unitary Matrix Models and Painlevé III

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Abstract

We discussed the full unitary matrix models from the view points of integrable equations and string equations. Coupling the Toda equations and the string equations, we derive a special case of the Painlevé III equation. From the Virasoro constrains, we can use the radial coordinate. The relation between t_1 and t_{-1} is like the complex conjugate.

1 Introduction

Models of the symmetric unitary matrix model are solved exactly in the double scaling limit, using orthogonal polynomials on a circle.[1] The partition function is the form $\int dU \exp\{-\frac{N}{\lambda}\mathrm{tr}V(U)\}$, where U is an $N \times N$ unitary matrix and $\mathrm{tr}V(U)$ is some well defined function of U. When $V(U)$ is the self adjoint we call the model symmetric.[2] The simplest case is given by $V(U) = U + U^{\dagger}$. This unitary models has been studied in connection with the large-N approximation to QCD in two dimensions.(*one-plaquette model*)[3] For this model "string equation" is the Painlevé II equation. "Discrete string equation" is called the discrete Painlevé II equation.[4] When $V(U)$ is the anti-self adjoint, we call the model anti-symmetric model. The simplest case is given by $V(U) = U - U^{\dagger}$. This is the theta term in the two-dimensional QCD.[5] It has the topological meaning. The full non-reduced unitary model was first discussed in [6]. The full unitary model can be embedded in the two-dimensional Toda Lattice hierarchy.

In this letter we shall try to reformulate the full unitary matrix model from the view points of integrable equations and string equations. These two view points are closely connected each other to describe this model. We unify these view points and clarify a relation between these view points.

This letter is organized as follows. In the section 2 we present discrete string equations for the full unitary matrix model. Here we consider only the simplest case. From the Virasoro constraints, a relation between times t_1 and t_{-1} is like complex conjugate. Because of this symmetry, we can use the radial coordinate. In the section 3 coupling the Toda equation and the discrete string equations, we obtain the special Painlevé III equation. In the section 4 we consider the reduced models, the symmetric and the anti-symmetric model. From the symmetric and the anti-symmetric model we can obtain the modified Volterra equation and the discrete nonlinear Schrödinger equation respectively. We study the relation of the symmetric and the anti-symmetric model. In a special case, we can transform the symmetric model into the anti-symmetric model. Using this map, we can obtain Bäcklund transformation of the modified Volterra equation and the discrete nonlinear Schrödinger equation. The last section is devoted to concluding remarks.

2 Unitary Matrix model

It is well known that the partition function τ_n of the unitary matrix model can be presented as a product of norms of the biorthogonal polynomial system. Namely, let us introduce a scalar product of the form

$$< A, B >= \oint \frac{d\mu(z)}{2\pi i z} A(z)B(z^{-1}). \qquad (2.1)$$

where

$$d\mu(z) = dz \exp\{-\sum_{m>0}(t_m z^m + t_{-m} z^{-m})\} \tag{2.2}$$

Let us define the system of the polynomials biorthogaonal with respect to this scalar product

$$< \Phi_n, \Phi_k^* >= h_k \delta_{nk}. \tag{2.3}$$

Then, the partition function τ_n of the unitary matrix model is equal to the product of h_n's:

$$\tau_n = \prod_{k=0}^{n-1} h_k, \quad \tau_0 = 1. \tag{2.4}$$

The polynomials are normalized as follows (we should stress that superscript '*' does not mean the complex conjugation):

$$\Phi_n = z^n + \cdots + S_{n-1}, \;\; \Phi_n^* = z^n + \cdots + S_{n-1}^*, \;\; S_{-1} = S_{-1}^* \equiv 1. \tag{2.5}$$

Now it is easy to show that these polynomials satisfy the following recurrent relations,

$$\begin{aligned} \Phi_{n+1}(z) &= z\Phi_n(z) + S_n z^n \Phi_n^*(z^{-1}), \\ \Phi_{n+1}^*(z^{-1}) &= z^{-1}\Phi_n^*(z^{-1}) + S_n^* z^{-n}\Phi_n(z), \end{aligned} \tag{2.6}$$

and

$$\frac{h_{n+1}}{h_n} = 1 - S_n S_n^*. \tag{2.7}$$

Note that h_n, S_n, S_n^* ,$\Phi_n(z)$ and Φ_n^* depend parametrically on $t_1, t_2, \cdots$, and $t_{-1}, t_{-2}, \cdots$, but for convenience of notation we suppress this dependence. Hereafter we call $t_1, t_2, \cdots$, and $t_{-1}, t_{-2}, \cdots$, time variables.

Using (2.3) and integration by parts, we can obtain next relations:

$$\begin{aligned} &- \oint \frac{d\mu(z)}{2\pi i z} V'(z)\Phi_{n+1}(z)\Phi_n^*(z^{-1}) \\ = & -\oint \frac{d\mu(z)}{2\pi i z}\frac{\partial \Phi_{n+1}(z)}{\partial z}\Phi_n^*(z^{-1}) - \oint \frac{d\mu(z)}{2\pi i z}\Phi_{n+1}(z)\frac{\partial \Phi_n^*(z^{-1})}{\partial z} + \oint \frac{d\mu(z)}{2\pi i z}\frac{\Phi_{n+1}(z)\Phi_n^*}{z} \\ = & \; (n+1)(h_{n+1} - h_n), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} & \oint \frac{d\mu(z)}{2\pi i z} z^2 V'(z)\Phi_{n+1}^*(z^{-1})\Phi_n(z) \\ = & \oint \frac{d\mu(z)}{2\pi i z} z^2 \frac{\partial \Phi_{n+1}^*(z^{-1})}{\partial z}\Phi_n(z) + \oint \frac{d\mu(z)}{2\pi i z} z^2 \Phi_{n+1}^*(z^{-1})\frac{\partial \Phi_n(z)}{\partial z} + \oint \frac{d\mu(z)}{2\pi i z} z\Phi_{n+1}^*(z^{-1})\Phi_n(z) \\ = & \; (n+1)(h_{n+1} - h_n). \end{aligned} \tag{2.9}$$

(2.8) and (2.9) are string equations for the full unitary matrix model.

If t_1 and t_{-1} are free variables while $t_2 = t_3 = \cdots = 0$ and $t_{-2} = t_{-3} = \cdots = 0$, (2.8) and (2.9) become

$$(n+1)S_n S_n^* = t_{-1}(S_n S_{n+1}^* + S_n^* S_{n-1})(1 - S_n S_n^*), \tag{2.10}$$

$$(n+1)S_n S_n^* = t_1(S_n^* S_{n+1} + S_n S_{n-1}^*)(1 - S_n S_n^*). \tag{2.11}$$

Next we introduce a useful relation. Using (2.3) and integration by parts, we can show

$$\begin{aligned} & \oint \frac{d\mu(z)}{2\pi i z} z V'(z) \Phi_n(z) \Phi_(n)^*(z^{-1}) \\ = & \oint \frac{d\mu(z)}{2\pi i z} z \frac{\partial \Phi_n(z)}{\partial z} \Phi_n^*(z^{-1}) + \oint \frac{d\mu(z)}{2\pi i z} z \Phi_n(z) \frac{\partial \Phi_n^*(z^{-1})}{\partial z} \\ = & \; n h_n - n h_n = 0. \end{aligned} \tag{2.12}$$

This corresponds to the Virasoro constraint:[2]

$$L_0^{-cl} = \sum_{k=-\infty}^{\infty} k t_k \frac{\partial}{\partial t_n}. \tag{2.13}$$

This relation constrains a symmetry like complex conjugate between t_k and t_{-k}. It is important in the next section. If we set that t_1 and t_{-1} are free variables while $t_2 = t_3 = \cdots = 0$ and $t_{-2} = t_{-3} = \cdots = 0$, from (2.12) we get

$$t_1 S_n S_{n-1}^* = t_{-1} S_n^* S_{n-1}. \tag{2.14}$$

Using (2.14), (2.10) and (2.11) can be written

$$(n+1)S_n = (t_1 S_{n+1} + t_{-1} S_{n-1})(1 - S_n S_n^*), \tag{2.15}$$

$$(n+1)S_n^* = (t_{-1} S_{n+1}^* + t_1 S_{n-1}^*)(1 - S_n S_n^*). \tag{2.16}$$

3 Toda equation and String equations

Using the orthogonal conditions, it is also possible to obtain the equations which describe the time dependence of $\Phi_n(z)$ and $\Phi_n^*(z)$. Namely, differentiating (2.3) with respect to times t_1 and t_{-1} gives the following evolution equations:

$$\frac{\partial \Phi_n(z)}{\partial t_1} = -\frac{S_n}{S_{n-1}} \frac{h_n}{h_{n-1}} (\Phi_n(z) - z\Phi_{n-1}), \tag{3.1}$$

$$\frac{\partial \Phi_n(z)}{\partial t_{-1}} = \frac{h_n}{h_{n-1}} \Phi_{n-1}(z), \tag{3.2}$$

$$\frac{\partial \Phi_n^*(z^{-1})}{\partial t_1} = \frac{h_n}{h_{n-1}} \Phi_{n-1}^*(z^{-1}), \tag{3.3}$$

$$\frac{\partial \Phi_n^*(z^{-1})}{\partial t_{-1}} = -\frac{S_n^*}{S_{n-1}^*}\frac{h_n}{h_{n-1}}(\Phi_n^*(z^{-1}) - z^{-1}\Phi_{n-1}^*), \tag{3.4}$$

The compatibility condition gives the following nonlinear evolution equations:

$$\frac{\partial S_n}{\partial t_1} = -S_{n+1}\frac{h_{n+1}}{h_n}, \quad \frac{\partial S_n}{\partial t_{-1}} = S_{n-1}\frac{h_{n+1}}{h_n}, \tag{3.5}$$

$$\frac{\partial S_n^*}{\partial t_1} = S_{n+1}^*\frac{h_{n+1}}{h_n}, \quad \frac{\partial S_n^*}{\partial t_{-1}} = -S_{n-1}^*\frac{h_{n+1}}{h_n}, \tag{3.6}$$

$$\frac{\partial h_n}{\partial t_1} = S_n S_{n-1}^* h_n, \quad \frac{\partial h_n}{\partial t_{-1}} = S_n^* S_{n-1} h_n, \tag{3.7}$$

Here we define a_n, b_n and b_n^*:

$$a_n \equiv 1 - S_n S_n^* = \frac{h_{n+1}}{h_n}, \tag{3.8}$$

$$b_n \equiv S_n S_{n-1}^*, \tag{3.9}$$

$$b_n^* \equiv S_n^* S_{n-1}. \tag{3.10}$$

Notice that from the definitions a_n, b_n and b_n^* satisfy the following identity:

$$b_n b_n^* = (1 - a_n)(1 - a_{n-1}). \tag{3.11}$$

It can be shown using (2.14) that

$$t_1 b_n = t_{-1} b_n^*. \tag{3.12}$$

In terms of a_n, b_n and b_n^*, (3.5) and (3.6) become the two-dimensional Toda equations:

$$\frac{\partial a_n}{\partial t_1} = a_n(b_{n+1} - b_n), \quad \frac{\partial b_n}{\partial t_{-1}} = a_n - a_{n-1}, \tag{3.13}$$

and

$$\frac{\partial a_n}{\partial t_{-1}} = a_n(b_{n+1}^* - b_n^*), \quad \frac{\partial b_n^*}{\partial t_1} = a_n - a_{n-1}. \tag{3.14}$$

Using a_n, b_n and b_n^*, we rewrite (2.10) and (2.11)

$$\frac{n+1}{t_{-1}}\frac{1-a_n}{a_n} = b_{n+1}^* + b_n^*. \tag{3.15}$$

and

$$\frac{n+1}{t_1}\frac{1-a_n}{a_n} = b_{n+1} + b_n, \tag{3.16}$$

From (3.13) and (3.16) we eliminate b_{n+1},

$$2b_n = \frac{1}{a_n}[\frac{n+1}{t_1}(1-a_n) - \frac{\partial a_n}{\partial t_1}]. \tag{3.17}$$

In the same way, from (3.14) and (3.15) we eliminate b^*_{n+1},

$$2b^*_n = \frac{1}{a_n}[\frac{n+1}{t_{-1}}(1-a_n) - \frac{\partial a_n}{\partial t_{-1}}]. \tag{3.18}$$

Using (3.11) and (3.12), (3.13) and (3.14) can be written

$$\frac{\partial b_n}{\partial t_{-1}} = (a_n - 1) + \frac{t_1}{t_{-1}}\frac{b_n^2}{1-a_n}, \tag{3.19}$$

$$\frac{\partial b^*_n}{\partial t_1} = (a_n - 1) + \frac{t_{-1}}{t_1}\frac{(b^*_n)^2}{1-a_n}. \tag{3.20}$$

Using (3.17) and (3.19) to eliminate b_n we obtain a second order ODE for a_n

$$\begin{aligned}\frac{\partial^2 a_n}{\partial t_1 \partial t_{-1}} &= \frac{n+1}{t_{-1}a_n}\frac{\partial a_n}{\partial t_1} - \frac{n+1}{t_1 a_n}\frac{\partial a_n}{\partial t_{-1}} - 2a_n(a_n-1) + \frac{(n+1)^2}{2t_1 t_{-1}}\frac{a_n - 1}{a_n} \\ &+ \frac{1}{a_n}\frac{\partial a_n}{\partial t_1}\frac{\partial a_n}{\partial t_{-1}} + \frac{1}{2}\frac{t_1}{t_{-1}}\frac{1}{(a_n-1)a_n}(\frac{\partial a_n}{\partial t_1})^2. \end{aligned} \tag{3.21}$$

In the same way, we eliminate b^*_n using (3.18) and (3.19) and obtain an ODE for a_n

$$\begin{aligned}\frac{\partial^2 a_n}{\partial t_1 \partial t_{-1}} &= \frac{n+1}{t_1 a_n}\frac{\partial a_n}{\partial t_{-1}} - \frac{n+1}{t_{-1} a_n}\frac{\partial a_n}{\partial t_1} - 2a_n(a_n-1) + \frac{(n+1)^2}{2t_1 t_{-1}}\frac{a_n - 1}{a_n} \\ &+ \frac{1}{a_n}\frac{\partial a_n}{\partial t_1}\frac{\partial a_n}{\partial t_{-1}} + \frac{1}{2}\frac{t_{-1}}{t_1}\frac{1}{(a_n-1)a_n}(\frac{\partial a_n}{\partial t_{-1}})^2. \end{aligned} \tag{3.22}$$

The equality of (3.21) and (3.22) implies that

$$t_1\frac{\partial a_n}{\partial t_1} = t_{-1}\frac{\partial a_n}{\partial t_{-1}} \tag{3.23}$$

Also this constraint can be shown from (2.14), (3.13) and (3.14) directly. So a_n are functions of the radial coordinate

$$x = t_1 t_{-1}, \tag{3.24}$$

only. Then from (3.21) and (3.22) we can obtain

$$\frac{\partial^2 a_n}{\partial x^2} = \frac{1}{2}(\frac{1}{a_n - 1} + \frac{1}{a_n})(\frac{\partial a_n}{\partial x})^2 - \frac{1}{x}\frac{\partial a_n}{\partial x} - \frac{2}{x}a_n(a_n-1) + \frac{(n+1)^2}{2x^2}\frac{a_n-1}{a_n}. \tag{3.25}$$

This is an expression of the Painlevé V equation (PV) with

$$\alpha_V = 0, \quad \beta_V = -\frac{(n+1)^2}{2}, \quad \gamma_V = 2, \quad \delta_V = 0. \tag{3.26}$$

(3.25) is related to the usual one through

$$a_n \longrightarrow c_n = \frac{a_n}{a_n - 1}. \tag{3.27}$$

(3.25) is the Painlevé III equation (P III) with (see [7])

$$\alpha_{III} = 4(n+1) \quad \beta_{III} = -4n, \quad \gamma_{III} = 4, \quad \delta_{III} = -4. \tag{3.28}$$

4 Symmetric and Anti-symmetric model

In this section we consider reduced unitary matrix models. The following reductions of the time variables t_k leads to the symmetric and the anti-symmetric model:

$$t_k = t_{-k} = t_k^+ \quad k = 1, 2, \cdots, \quad (\mathrm{symmetric\ \ model}) \tag{4.1}$$

and

$$t_k = -t_{-k} = t_k^- \quad k = 1, 2, \cdots, \quad (\mathrm{anti-symmetric\ \ model}) \tag{4.2}$$

If t_1^+ are free variables while $t_2^+ = t_3^+ = \cdots = 0$, from (2.14) $S_n = S_n^*$. From (2.12) and (2.14) the string equation becomes

$$(n+1)S_n = t_1^+(S_{n+1} + S_{n-1})(1 - S_n^2). \tag{4.3}$$

This is called the discrete Painlevé II (dP II) equation. Appropriate continuous limit of (4.3) yields the Painlevé II (P II) equation. From (3.5) and (3.6) we can obtain the modified Volterra equation:[6]

$$\frac{\partial S_n}{\partial t_1^+} = -(1 - S_n^2)(S_{n+1} - S_{n-1}). \tag{4.4}$$

Appropriate continuous limit of (4.4) yields the modified KdV equation.

(4.3) and (4.4) can be written in the form

$$2S_{n+1} = \frac{1}{1 - S_n^2}\left(\frac{n+1}{t_1^+}S_n - \frac{\partial S_n}{\partial t_1^+}\right), \tag{4.5}$$

and

$$2S_{n-1} = \frac{1}{1 - S_n^2}\left(\frac{n+1}{t_1^+}S_n + \frac{\partial S_n}{\partial t_1^+}\right), \tag{4.6}$$

Writing (4.6) as

$$2S_n = \frac{1}{1 - S_{n+1}^2}\left(\frac{n+2}{t_1^+}S_{n+1} + \frac{\partial S_{n+1}}{\partial t_1^+}\right), \tag{4.7}$$

and using (4.5) to eliminate S_{n+1} we obtain a second order ODE for S_n.

$$\frac{\partial^2 S_n}{\partial t_1^{+2}} = -\frac{S_n}{1 - S_n^2}\left(\frac{\partial S_n}{\partial t_1^+}\right)^2 - \frac{1}{t_1^+}\frac{\partial S_n}{\partial t_1^+} + \frac{(n+1)^2}{t_1^{+2}}\frac{S_n}{1 - S_n^2} - 4S_n(1 - S_n^2). \tag{4.8}$$

It is important to keep in mind that the relevant function is $1 - S_n^2 = a_n$. a_n satisfies

$$\frac{\partial^2 a_n}{\partial t_1^{+2}} = \frac{1}{2}\left(\frac{1}{a_n - 1} + \frac{1}{a_n}\right)\left(\frac{\partial a_n}{\partial t_1^+}\right)^2 - \frac{1}{t_1^+}\frac{\partial a_n}{\partial t_1^+} - 8a_n(a_n - 1) + \frac{2(n+1)^2}{t_1^{+2}}\frac{a_n - 1}{a_n}. \tag{4.9}$$

If we set $x = (t_1^+)^2$, (4.9) is the same as (3.25). Then, we obtain (4.9) which is the special case of Painlevé III. In conclusion, coupling the modified Volterra and the dP II, we can obtain the P III. The double limit

$$n \to \infty, \quad t_1^+ \to \infty, \quad \frac{t_1^{+2}}{n} = O(1), \tag{4.10}$$

maps P III(4.9) to P II. Clearly, this kind of limit can be discussed independently of the connection with the modified Volterra and the modified KdV equation.

Next we consider the anti-symmetric model. If t_1^- are free variables while $t_2^- = t_3^- = \cdots = 0$, from (2.14)

$$S_n S_{n-1}^* = S_n^* S_{n-1}. \tag{4.11}$$

From (2.12) and (2.14) the string equations become

$$\begin{aligned} (n+1)S_n &= t_1^-(-S_{n+1} + S_{n-1})(1 - S_n S_n^*), \\ (n+1)S_n^* &= t_1^-(S_{n+1}^* - S_{n-1}^*)(1 - S_n S_n^*). \end{aligned} \tag{4.12}$$

On the other hand, from (3.5) and (3.6) we obtain the discrete nonlinear Schrödinger (NLS) equation:[8]

$$\begin{aligned} \frac{\partial S_n}{\partial t_1^-} &= -(1 - S_n S_n^*)(S_{n+1} + S_{n-1}), \\ \frac{\partial S_n^*}{\partial t_1^-} &= (1 - S_n S_n^*)(S_{n+1}^* + S_{n-1}^*). \end{aligned} \tag{4.13}$$

Using the same method in the symmetric model case we can obtain the P III. Coupling the discrete NLS and the string equation, we can obtain P III.

Through the transformation

$$z \to iz \quad it_1^- \to t_1^+, \tag{4.14}$$

the anti-symmetric model is transformed into the symmetric model. Then we get the Bäcklund transformation from the discrete NLS to the modified Volterra equation:

$$S_n \longrightarrow (i)^{n+1} S_n. \tag{4.15}$$

However we restrict t_1^- to a real number, we can not transform the anti-symmetric model into the symmetric model.

We change variables $a_n \to u_n = \ln a_n$. Then (3.13) and (3.14) become

$$\frac{\partial^2 u_n}{\partial t_1 \partial t_{-1}} = e^{u_{n+1}} - 2e^{u_n} + e^{u_{n-1}}. \tag{4.16}$$

In the anti-symmetric model from (2.5) and (4.11) we can get

$$\begin{aligned} S_n &= S_n^*, \quad a_n = 1 - S_n^2, \qquad (n = \text{odd}), \\ S_n &= -S_n^*, \quad a_n = 1 + S_n^2, \qquad (n = \text{even}). \end{aligned} \tag{4.17}$$

In the case that t_1^- is real, we can see the oscillation of a_n. This phenomenon can be seen only in the anti-symmetric model.

Here we consider the continuum limit near the anti-symmetric model. We are interested in $S_n^2 = \epsilon g_n$, $\epsilon \to 0$ and $n \to \infty$. We assume $t_1 = -t_{-1} + 2\epsilon/n$ and define $g_{n+1} - g_n = \epsilon g_n'$.

Then the continuum limit yields

$$u \equiv u_n = -u_{n+1} + \epsilon(\pm g_n' + g_n^2) + O(\epsilon^2), \tag{4.18}$$

where $\pm$ corresponds to n =odd and n =even respectively. So in the continuous limit (4.16) becomes well known the 1D sinh Gordon equation

$$\frac{\partial^2 u}{\partial t_1 \partial t_{-1}} = -2 \sinh 2u, \tag{4.19}$$

where $t_1 = -t_{-1}$. Here we introduce the radial coordinate

$$r = \sqrt{-t_1 t_{-1}}. \tag{4.20}$$

u obeys an ODE of the form

$$\frac{\mathrm{d}^2 u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} = 2 \sinh 2u. \tag{4.21}$$

This is the P III with

$$\alpha_{III} = 0 \quad \beta_{III} = 0, \quad \gamma_{III} = 1, \quad \delta_{III} = -1. \tag{4.22}$$

This equation is obtained from the 2 states Toda field equation, too.[9] Because of the oscillation of a_n, in the continuous limit u_n looks like having 2 states.

At last we consider the relation between the symmetric and anti-symmetric model from the determinant form. The partition function of the symmetric model is

$$\tau_N^+ = \det_{ij} I_{i-j}(t_1^+), \tag{4.23}$$

where I_m is the modified Bessel function of order m.[10] In the same way, we can calculate the partition function of the anti-symmetric model:

$$\tau_N^- = \det_{ij} J_{i-j}(t_1^-), \tag{4.24}$$

where J_m is the Bessel function of order m. (4.14) is also the transformation between the Bessel and the modified Bessel function. (4.17) comes from the oscillation of the Bessel function.

5 Concluding remarks

We try to reformulate the full unitary matrix model from the view points of integrable equations and string equations. Coupling the Toda equation and the string equations, we obtain the P III equation. Because of the Virasoro constraint, t_1 and t_{-1} have the symmetry. This symmetry is like complex conjugate. Then we can use the radial coordinate. This PIII also describe the phase transition between the week and strong coupling region. Next we consider the relation among the symmetric, anti-symmetric model and the P III equation. If t_1^- is a purely imaginary number, the anti-symmetric model can be transformed into the anti-symmetric model. Using this map we construct the Bäcklund transformation from the discrete nonlinear Schrödinger equation to the modified Volterra equation . This map is also the transformation between the Bessel and the modified Bessel function. If we restrict t_1^- to a real number, the symmetric and the anti-symmetric are different.

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